APPLICATION OF A VARIATIONAL PRINCIPLE TO INVESTIGATE DISCONTINUITIES IN A CONTINUUM

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The question of the application of a variational principle as the fundamental basis for the construction of models of media within the scope of the special or general theories of relativity was considered in detail by Sedov [1 and 2]. In the foregoing work variational principles are applied to obtain conditions on the surfaces of discontinuity of the characteristics of the medium. Equations are thus obtained on discontinuities in media having internal degrees of freedom, as is expressed in the presence of the strain rate tensors and strain gradients among their governing parameters. Such media have been considered, for example, in [3 to 9].

The question of boundary conditions and relationships on the discontinuities has been investigated in [10 to 14] by other methods for different media with microstructure.

The analysis is conducted within the scope of Newtonian mechanics (*).

1. Variational principle. Let us consider an arbitrarily isolated volume of a continuum $V(\xi^1, \xi^2, \xi^3, t)$ referred to the Lagrangian coordinates ξ^1, ξ^2, ξ^3 . Following [3], let us introduce three reference systems for the motion characteristics.

1. A moving Lagrangian system with basis $\partial_i{}^\wedge$ and metric tensor (the aotual space)

$$G^\wedge = g_{ij}\partial^{\wedge i}\partial^{\wedge j}$$

2. A fixed Lagrangian system with basis $\partial_i{}^\circ$ and metric tensor (the space of initial states)

$$G^\circ = g_{ij}{}^\circ\partial^{i\circ}\partial^{j\circ}$$

3. The fixed system of the observer $\partial_i(x^1, x^2, x^3)$, with respect to which the motion is considered (we shall consider the ∂_i as Cartesian system)

$$x^k = x^k(\xi^1, \xi^2, \xi^3, t), \qquad \mathbf{r} = \mathbf{r}_0 + \mathbf{u}(\xi^1, \xi^2, \xi^3, t)$$

Here $\mathbf{u}$ is the displacement vector of the medium particles; $\mathbf{v}$ the velocity vector of the medium particles

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*) This limitation is not too essential. Methods developed in the mentioned works [1 and 2] permit easy extension of the obtained deductions to the case of the special and general theories of relativity.

$$\mathbf{u} = u^i\ni_i = u^{\wedge i}\ni_i^{\wedge}, \qquad \mathbf{v} = v^i\ni_i = v^{\wedge i}\ni_i^{\wedge}, \qquad v^i = \frac{\partial u^i}{\partial t}_{\xi^k=\mathrm{const}}$$

We shall utilize variational equations of the form [1]

$$\delta\int_{t_0}^{t_1}\int_V L d\tau dt + \delta W + \delta W^* = 0 \tag{1.1}$$

as the initial basis for construction of the model of our medium.

Here L is a Lagrange function dependent, according to the manner admitted by invariance considerations, on v^i, the initial density ρ_0, the entropy $-S$, g_{ij}°, the tensor g_{ij}, its time derivative $g_{ij}^\cdot$, gradients with respect to the initial space (*)

$$\nabla_k^\circ g_{ij} = \frac{\partial g_{ij}}{\partial \xi^k} - \Gamma^{\circ\omega}_{ki} g_{\omega j} - \Gamma^{\circ\omega}_{kj} g_{\omega i}$$

so that

$$L = L(\xi^k, \rho_0, g_{ij}^\cdot\ v^i, g_{ij}, g_{ij}^\cdot, \nabla_k^\circ g_{ij}, S)$$

The variation δW (δ for constant Lagrangian coordinates) is an integral over the surface $\Sigma(\xi^k, t)$ bounding the volume $V(\xi^k, t)$, of the linear combination δu^ω, δg_{ij} and is determined by the assignment of L; δW^* is given and taken as

$$\int_{t_0}^{t_1}\int_V \rho T \delta S\, d\tau dt$$

The expression of the variational principle may be given another form if the integration is carried out over the fixed volume V_0 and its surface Σ_0 in the space of initial states, i.e. over the prototype of the volume $V(\xi^1, \xi^2, \xi^3, t)$ at the initial instant. To do this let us introduce the Jacobian of the transformation

$$\Delta = \det\left\|\frac{\partial x^i}{\partial \xi^j}\right\| = \frac{\sqrt{g}}{\sqrt{g_0}} = \left(\frac{\det\|g_{ik}\|}{\det\|\overset{\circ}{g}_{ik}\|}\right)^{1/2}$$

$$d\tau = \sqrt{g}\, d\xi^1 d\xi^2 d\xi^3, \quad d\tau_0 = \sqrt{g_0}\, d\xi^1, d\xi^2, d\xi^3, \ d\tau_\xi = d\xi^1 d\xi^2 d\xi^3$$

Then

$$\delta\int_{t_0}^{t_1}\int_{V_0} L\sqrt{g}\, d\tau_\xi dt + \delta W + \int_{t_0}^{t_1}\int_{V_0} \rho\sqrt{g}\, T\delta S d\tau_\xi dt = 0 \tag{1.2}$$

In evaluating the variations in (1.2) it is necessary to take account of the following relationships:

$$\delta v^i = \frac{\partial}{\partial t}(\delta u^i)_{\xi^k=\mathrm{const}}, \qquad \delta\rho_0 = 0, \qquad \delta g_{ij}^\circ = 0$$

$$\delta g_{ij} = \delta(\ni_i^\wedge \ni_j^\wedge) = \ni_i^\wedge \delta\ni_j^\wedge + \ni_j^\wedge\delta\ni_i^\wedge = \ni_i^\wedge\delta\frac{\partial \mathbf{r}}{\partial\xi^j} + \ni_j^\wedge\delta\frac{\partial\mathbf{r}}{\partial\xi^i} =$$

$$= \ni_i^\wedge\frac{\partial}{\partial\xi^j}(\delta u_k^\wedge \ni^{\wedge k}) + \ni_j^\wedge\frac{\partial}{\partial\xi^i}(\delta u_k^\wedge\ni^{\wedge k}) = \nabla_j^\wedge\delta u_i^\wedge + \nabla_i^\wedge\delta u_j^\wedge$$

$$\nabla_j^\wedge u_i^\wedge = \frac{\partial u_i^\wedge}{\partial\xi^j} - \Gamma^{\wedge\omega}_{ij} u_\omega^\wedge, \qquad \delta u_i^\wedge = g_{ik}\frac{\partial\xi^k}{\partial x^\omega}\delta u^\omega$$

$$\delta\nabla_k^\circ g_{ij} = \nabla_k^\circ\delta g_{ij}, \quad \delta g_{ij}^\cdot = \frac{\partial}{\partial t}(\delta g_{ij})_{\xi^k=\mathrm{const}}$$

Here and henceforth it is considered that the variations δu^i are

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*) It would be possible to consider $\nabla_k^\wedge g_{ij}^\circ$ the derivative with respect to the actual space instead of $\nabla_k^\circ g_{ij}$. However, by virtue of the existing interrelationships [4 and 5], this would only result in a definition of the function L.

continuous functions with continuous derivatives with respect to ξ^k and t to second order, inclusive. In the absence of higher derivatives among the governing parameters it is sufficient to require just the existence of first order derivatives.

It is easy also to verify the validity of Formula

$$A^{ij}\delta g_{ij} = \frac{\partial}{\partial \xi^j}\left(2A^{ij} g_{ik} \frac{\partial \xi^k}{\partial x^\omega}\delta u^\omega\right) - \sqrt{g}\, g_{ik}\frac{\partial \xi^k}{\partial x^\omega}\nabla_j{}^\wedge \left(\frac{2}{\sqrt{g}} A^{ij}\right)\delta u^\omega$$

Performing the variation in (1.2), we obtain after the customary manipulation

$$\begin{aligned} -\int_{t_0}^{t_1}\int_{V_0}\Big\{\frac{\partial}{\partial t}\frac{\partial L\sqrt{g}}{\partial v^\omega} + \sqrt{g}\, g_{ik}\frac{\partial \xi^k}{\partial x^\omega}\nabla_j{}^\wedge\Big[\frac{2}{\sqrt{g}}\frac{\partial L\sqrt{g}}{\partial g_{ij}} - 2\frac{\sqrt{g_0}}{\sqrt{g}}\nabla_k{}^\circ \frac{1}{\sqrt{g_0}}\frac{\partial L\sqrt{g}}{\partial \nabla_k{}^\circ g_{ij}} - \\ - \frac{2}{\sqrt{g}}\frac{\partial}{\partial t}\frac{\partial L\sqrt{g}}{\partial \dot{g}_{ij}}\Big]\Big\}\delta u^\omega d\tau_\xi dt + \int_{t_0}^{t_1}\int_{V_0}\left(\frac{\partial L\sqrt{g}}{\partial S} - \rho\sqrt{g}T\right)\delta S d\tau_\xi dt + \\ + \int_{t_0}\int_{\Sigma_0}\Big\{\frac{\partial L\sqrt{g}}{\partial v^\omega} n_t\delta u^\omega + \frac{\partial L\sqrt{g}}{\partial \dot{g}_{ij}} n_t\delta g_{ij} + \frac{\partial L\sqrt{g}}{\partial \nabla_k{}^\circ g_{ij}}\delta g_{ij} n_k{}^\circ + \\ + 2g_{ik}\frac{\partial \xi^k}{\partial x^\omega}\left(\frac{\partial L\sqrt{g}}{\partial g_{ij}} - \sqrt{g_0}\nabla_k{}^\circ\frac{1}{\sqrt{g_0}}\frac{\partial L\sqrt{g}}{\partial \nabla_k{}^\circ g_{ij}} - \frac{\partial}{\partial t}\frac{\partial L\sqrt{g}}{\partial \dot{g}_{ij}}\right) n_j{}^\circ\Big\}\delta u^\omega d\sigma_0 dt + \delta W = 0 \end{aligned} \qquad (1.3)$$

Here the lower limits t_0 and Σ_0 denote integration over the four-dimensional space bounding the volume in the space of the ξ^k, t coordinates considered as Cartesian coordinates; the n_t, $n_j{}^\circ$ denote components of the unit vector normal to this surface. Because of the arbitrariness of the variation within and on the boundary of the region of integration, as well as of the region of integration itself, Equation (1.3) yields

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial L\sqrt{g}}{\partial v^\omega} + \sqrt{g}\, g_{ik}\frac{\partial \xi^k}{\partial x^\omega}\nabla_j{}^\wedge\Big(\frac{2}{\sqrt{g}}\frac{\partial L\sqrt{g}}{\partial g_{ij}} - 2\frac{\sqrt{g_0}}{\sqrt{g}}\nabla_k{}^\circ\frac{1}{\sqrt{g_0}}\frac{\partial L\sqrt{g}}{\partial \nabla_k{}^\circ g_{ij}} - \\ - \frac{2}{\sqrt{g}}\frac{\partial}{\partial t}\frac{\partial L\sqrt{g}}{\partial \dot{g}_{ij}}\Big) = 0 \end{aligned} \qquad (1.4)$$

The obtained equations should be considered as the equations of motion of the medium in Lagrangean form in projections on the axes of the observer's system $\ni_i$.

The surface integral leads to the relationship

$$\delta W = \iint_{t_0\Sigma_0}\left(\sqrt{g}\, p^{ij}\delta u_i{}^\wedge n_j{}^\circ - Q^{kij}\delta g_{ij} n_k{}^\circ - J_\omega \delta u^\omega n_t - J^{ij}\delta g_{ij} n_t\right) d\sigma_0\, dt \qquad (1.5)$$

Here

$$p^{ij} = -\frac{2}{\sqrt{g}}\frac{\partial L\sqrt{g}}{\partial g_{ij}} + 2\frac{\sqrt{g_0}}{\sqrt{g}}\nabla_k{}^\circ\frac{1}{\sqrt{g_0}}\frac{\partial L\sqrt{g}}{\partial \nabla_k{}^\circ g_{ij}} + \frac{2}{\sqrt{g}}\frac{\partial}{\partial t}\frac{\partial L\sqrt{g}}{\partial \dot{g}_{ij}} \qquad (1.6)$$

$$J_\omega = \frac{\partial L\sqrt{g}}{\partial v^\omega}, \qquad J^{ij} = \frac{\partial L\sqrt{g}}{\partial \dot{g}_{ij}}, \qquad Q^{kij} = \frac{\partial L\sqrt{g}}{\partial \nabla_k{}^\circ g_{ij}} \qquad (1.7)$$

Moreover, the coefficient for δS yields

$$\frac{\partial L\sqrt{g}}{\partial S} = -\rho\sqrt{g}T \qquad (1.8)$$

In this notation the equations of motion take the form

$$\frac{\partial J_\omega}{\partial t} - \sqrt{g}\frac{\partial \xi^k}{\partial x^\omega}\nabla_j{}^\wedge p^j{}_k = 0 \qquad (1.9)$$

and the relationships (1.5) to (1.9) may be considered as generalized equations of the state of the medium, and in particular, p^{ij} may be considered as the stress tensor.

In conclusion, let us note that if a variation of the time t is carried out in the fundamental relationship (1.1), it would then permit the energy equation to be obtained. In fact, let us consider the variation $t^* = t + \delta t$, where we take δt as an arbitrary constant. In such a variation, the term

$$\int_{t_0}^{t_1}\int_V N\delta t\, d\tau\, dt$$

which vanishes for $\delta t = 0$, should be added to δW^* and all the variations should be considered total, i.e.

$$\delta q = \delta q_{t=\mathrm{const}} + q^{\cdot}\delta t.$$

Then we obtain the energy equation

$$\frac{\partial}{\partial t}\left(L\sqrt{g} - J_\omega v^\omega - J^{ij} g_{ij}^{\cdot}\right) + \frac{\partial}{\partial \xi^k}\left(\frac{1}{\sqrt{g}}\, p^{ik} v_i^{\wedge} - Q^{kij} g_{ij}^{\cdot}\right) + \rho\sqrt{g}N = 0 \qquad (1.10)$$

as the additional equation.

Multiplying (1.4) by v^ω and adding to the last, we obtain an equation for the entropy S by virtue of relationships (1.5) to (1.9)

$$T\,\frac{\partial S}{\partial t} = N \qquad (1.11)$$

Here N may be considered as the energy influx to the particle.

2. Examples. Model of an elastic body. The finite strain tensor $\varepsilon_{ij} = {}^1/_2\,(g_{ij} - g_{ij}^\circ)$. is introduced as the characteristic of the medium. Let us assume that the Lagrange function has $\rho_0, g_{ij}^\circ, \varepsilon_{ij}, v^k, S$. as arguments. Then we have the following relationships:

$$p^{ij} = -\frac{1}{\sqrt{g}}\frac{\partial L\sqrt{g}}{\partial g_{ij}}, \qquad J^{ij} = 0, \qquad Q^{kij} = 0$$

If it is assumed that $L = {}^1/_2 \rho v^i v_i - \rho U(\varepsilon_{ij}, S)$, where U is the energy of unit mass, then

$$p^{ij} = \rho\,\frac{\partial U}{\partial \varepsilon_{ij}}, \qquad J_\omega = \rho\sqrt{g}\,v_\omega$$

Model of an ideal fluid. This model is obtained from the preceding one if it is considered that $L = L(\rho_0, g_{ij}^\circ, \sqrt{g}, v^k, S)$, i.e. that L does not depend on all the components of the metric tensor but only on its determinant g. (By virtue of the continuity equation $\sqrt{g} = \rho^{-1}\rho_0\sqrt{g_0}$.) Taking into account the well-known formula from analysis

$$\frac{\partial\sqrt{g}}{\partial g_{ij}} = \frac{1}{2}\sqrt{g}\,g^{ij}$$

we obtain for the stress tensor

$$p^{ij} = -\frac{\partial L\sqrt{g}}{\partial\sqrt{g}}\,g^{ij} = -pg^{ij}, \qquad p = \frac{\partial L\sqrt{g}}{\partial\sqrt{g}}$$

Here p is considered as the pressure. The formula shows that the contravariant components of the stress tensor form a spherical tensor.

Its mixed components equal $p^i{}_k = -p\delta^i{}_k$. If the expression $L = \frac{1}{2}\rho v^i v_i - \rho U(\rho, S)$ is taken as the Lagrange function, then we obtain $p = \rho^2\,\delta U/\delta\rho$ for p.

In this case the equations of motion become

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} + \frac{\partial p}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^i} = 0$$

Model of a medium characterized by the density and the time derivative of the density. We take g_{ij}°, $\sqrt{g}$, $(\sqrt{g})^\cdot$, v^k, S as the governing parameters for the Lagrange function. Taking account of the relationships

$$\frac{\partial \sqrt{g}}{\partial g_{ij}} = \frac{\sqrt{g}}{2} g^{ij}, \qquad \frac{\partial (\sqrt{g})^\cdot}{\partial g_{ij}}\bigg|_{g_{ij}^\cdot = \mathrm{const}} = \frac{(\sqrt{g})^\cdot}{2} g^{ij} + \frac{\sqrt{g}}{2} g^{\cdot ij}$$

we obtain

$$p^{ij} = -\left[\frac{\partial L \sqrt{g}}{\partial \sqrt{g}} - \frac{\partial}{\partial t}\frac{\partial L \sqrt{g}}{\partial (\sqrt{g})^\cdot}\right] g^{ij} = -pg^{ij}, \qquad p = \frac{\partial L \sqrt{g}}{\partial \sqrt{g}} - \frac{\partial}{\partial t}\frac{\partial L \sqrt{g}}{\partial (\sqrt{g})^\cdot}$$

Here p is the pressure. If $L = {}^1\!/_2 \rho v^i v_i - \rho U(\rho, \rho^\cdot, S)$, then

$$p = \rho^2 \frac{\partial U}{\partial \rho} - \frac{\partial}{\partial t}\left(\rho^2 \frac{\partial U}{\partial \rho^\cdot}\right)_S$$

Such a model of a medium may be utilized to describe an ideal incompressible fluid with bubbles changing their volume [9].

It should be noted that the model of a medium has been constructed herein under the assumption of no additional energy influx to the particle dq^{**} connected with the internal degrees of freedom [3]. Our expression contains such an influx, equal to

$$dq^{**} = \frac{1}{\rho} d\left(\rho^2 \rho^\cdot \frac{\partial U}{\partial \rho^\cdot}\right)$$

By suitable selection of δW^* it is possible to obtain the model of a medium considered in [9]. Finally, the obtained formulas permit the analysis of examples of a medium [5 and 6] being characterized by space derivatives with respect to the density $\nabla_k{}^\circ \rho$.

3. Discontinuities in a continuum. In a continuum let there be a surface on which its characteristics undergo discontinuity. To find the conditions which the values of these characteristics should satisfy on the surface of discontinuity, let us use the following variational principle:

$$\delta \int_{t_0}^{t_1} \int_V L(\rho_0, g_{ij}^\circ, g_{ij}, \nabla_k{}^\circ g_{ij}, g_{ij}^\cdot, v^k, S)\, d\tau\, dt = 0 \tag{3.1}$$

For simplicity it is here considered that

$$\delta W = 0, \qquad \delta W^* = 0 \quad \text{for} \quad \delta u\,|_\Sigma = 0, \qquad \delta g_{ij}\,|_\Sigma = 0$$

The Lagrange function may itself have a different form on both sides of the surface of discontinuity. (Let us note that in such cases δW^* may not be zero because of the additional internal energy sources on the surface of discontinuity). Hence, the subsequent results also refer to the case when the surface of discontinuity is the interface between two media, and in the case of a stationary discontinuity the conditions on it may be considered as boundary conditions.

The equation of the surface of discontinuity is not known beforehand, hence not only the medium characteristics, but also the surface of discontinuity S_{04} are subject to variation. Let the discontinuity occur in the surface S_{04} whose equation is $F(\xi^1, \xi^2, \xi^3, t) = 0$, dividing the four-dimensional volume V_{04} into two parts V_{0+} and V_{0-}. As the comparison surface, the variational position of the surface of discontinuity (Fig.1), let us take the surface (S_{04}) defined with the aid of the virtual displacements along the normal δl_{n_0}

$$\delta l_{n0} = \delta \xi^k n_k{}^\circ + \delta t n_t$$

$$n_k{}^\circ = F_{\xi k} \Big/ \sqrt{F_{\xi^1}{}^2 + F_{\xi^2}{}^2 + F_{\xi^3}{}^2 + F_t{}^2}, \qquad n_t = F_t \Big/ \sqrt{F_{\xi^1}{}^2 + F_{\xi^2}{}^2 + F_{\xi^3}{}^2 + F_t{}^2}$$

and let us consider the total variation of the functional (3.1) over the domain V_{0+}, say, by taking account of the domain itself in the variation. This variation is the principal linear part of the change in the functional during integration over the volume $V_{0+} + \Delta V_0$ and V_{0+}

$$\int_{V_{0+}+\Delta V_0} (L\sqrt{g})\,d\tau_\xi\,dt - \int_{V_{0+}} L\sqrt{g}\,d\tau_\xi\,dt = \int_{V_{0+}} \delta L\sqrt{g}\,d\tau_\xi\,dt + \int_{\Delta V_0} L\sqrt{g}\,d\tau_\xi\,dt + R$$

Here R are higher order quantities. It is easy to note that to the accuracy of higher order quantities, the integral over ΔV_0 may be written as an integral over the surface S_{04}

$$\int_{\Delta V_0} L\sqrt{g}\,d\tau_\xi\,dt = \int_{S_{04}} L\sqrt{g}\,\delta l_{n0}\,d\sigma_0 = \int_{S_{04}} L\sqrt{g}\,(\delta\xi^k n_k^\circ + \delta t n_t)\,d\sigma_0$$

The expression for the total variation of the functional over the volume V_{0+} will be

$$\delta\int_{V_{0+}} L\sqrt{g}\,d\tau_\xi\,dt = \int_{V_{0+}} \delta L\sqrt{g}\,d\tau_\xi\,dt + \int_{S_{04}} L\sqrt{g}\,(\delta\xi^k n_k^\circ + \delta t n_t)\,d\sigma_0$$

Taking account of Formulas (1.5) to (1.9) from Section 1, we have

$$\delta\int_{V_{0+}} L\sqrt{g}\,d\tau_\xi\,dt = -\int_{V_{0+}} \left(\frac{\partial J_\omega}{\partial t} - \sqrt{g}\,\frac{\partial\xi^k}{\partial x^\omega}\,\nabla_j{}^\wedge p^j{}_k\right)\delta u^\omega\,d\tau_\xi\,dt +$$

$$+ \iint_{tS_{03}} \left(J_\omega n_t - \sqrt{g}\,\frac{\partial\xi^k}{\partial x^\omega}\,p^j{}_k n_j^\circ\right)\{\delta u^\omega\}\,d\sigma_0\,dt +$$

$$+ \iint_{tS_0^3} (J^{ij}n_t + Q^{kij}n_k^\circ)\{\delta g_{ij}\}\,d\sigma_0\,dt + \iint_{tS_{04}} (L\sqrt{g}\,n_t\delta t + L\sqrt{g}\,\delta\xi^k n_k^\circ)\,d\sigma_0\,dt \qquad (3.2)$$

Here the brace { } denotes that the variations are taken for $\delta\xi^k = \delta t = 0$. The volume integral in the right-hand side vanishes because of the equations of motion of the medium.

For the subsequent transformations it is necessary to take into account that all the variations $\{\delta g_{ij}\}$ on the surface are not independent.

Only that part of them will be independent which is expressed in terms of the variations of the derivatives of the displacements with respect to the normal to the surface. In order to have only independent variations in (3.2), let us use the evident relationships [15]

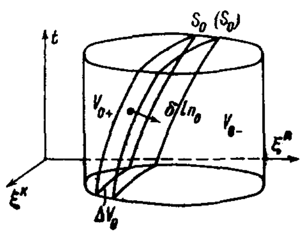

Fig. 1

$$\{\delta g_{ij}\} = \nabla_i{}^\wedge\{\delta u_j{}^\wedge\} + \nabla_j{}^\wedge\{\delta u_i{}^\wedge\}$$

$$\nabla_i{}^\wedge\{\delta u_j{}^\wedge\} = \frac{\partial\{\delta u_j{}^\wedge\}}{\partial\xi^i} - \Gamma^\wedge{}_{ij}^s\{\delta u_s{}^\wedge\}$$

$$\frac{\partial}{\partial\xi^\alpha} = (\delta_\alpha{}^k - n_\alpha n^k)\,\frac{\partial}{\partial\xi^k} + n_\alpha n^k\,\frac{\partial}{\partial\xi^k}$$

Here $n_k = n_k^\circ / \sqrt{n_1^{\circ 2} + n_2^{\circ 2} + n_3^{\circ 2}}$ are the components of the unit vector normal to the surface S_{03}

$$(\delta_\alpha{}^k - n_\alpha n^k)\,\frac{\partial}{\partial\xi^k} = D_\alpha, \qquad n_\alpha n^k\,\frac{\partial}{\partial\xi^k} = \frac{\partial}{\partial n}$$

are derivatives along the surface and with respect to the normal to the surface.

The following Formula [16] is valid in the notation accepted:

$$\int_{S_{03}} \Phi^{ij}\,\frac{\partial\{\delta u_j\}}{\partial\xi^i}\,d\sigma_0 = \int_{S_{03}} (n_i D_\alpha n^\alpha - D_i)\,\Phi^{ij}\{\delta u_j\}\,d\sigma_0 + \int_{S_{03}} \Phi^{ij}n_i\,\frac{\partial\{\delta u_j\}}{\partial n}\,d\sigma_0 \qquad (3.3)$$

Here S_{03} is a closed, smooth surface (*)(see this footnote on the next page). Taking the expression $2(J^{ij}n_t + Q^{kij}n_k^\circ)$ as Φ^{ij}, let us rewrite the variation over the volume V_{0+} as follows:

$$\delta\int\limits_{V_{0+}} L\sqrt{g}\,d\tau_\xi\,dt=\int\limits_t\int\limits_{S_{03}}\left[J_\omega n_t-\frac{\partial\xi^k}{\partial x^\omega}\left(\sqrt{g}p^j{}_k n_j^\circ-\Omega_{ijk}\Phi^{ij}\right)\right]\{\delta u^\omega\}\,d\sigma_0\,dt+$$
$$+\int\limits_t\int\limits_{S_{03}}\Phi^{ij}n_i\frac{\partial}{\partial n}\{\delta u_j{}^\wedge\}\,d\sigma_0\,dt+\int\limits_t\int\limits_{S_{03}}\left(L\sqrt{g}n_t\delta t+L\sqrt{g}\delta\xi^k n_k^\circ\right)d\sigma_0\,dt \qquad (3.4)$$
$$\Omega_{ijk}=(n_iD_\alpha n^\alpha-D_i)\,g_{jk}-\Gamma^\wedge{}_{ijk}$$

An analogous expression is obtained for the variations over the volume V_{0-}.

The expression for the sum of these variations, which equals zero, may be used to determine relationships on different kinds of discontinuities. Let us consider a discontinuity on which the displacements and their derivatives with respect to the normal remain continuous, so that

$$\delta u_+{}^\omega=\delta u_-{}^\omega,\qquad \frac{\partial}{\partial n}(\delta u_j{}^\wedge)_+=\frac{\partial}{\partial n}(\delta u_j{}^\wedge)_-$$

Then by putting $\delta\xi^k=\delta t=0$, we obtain the first group of relationships by virtue of the arbitrariness of the variations $\delta u_+{}^\omega$ and $\partial(\delta u_j{}^\wedge)/\partial n_+$

$$\left[J_\omega n_t-\frac{\partial\xi^k}{\partial x^\omega}\left(\sqrt{g}p^j{}_k n_j^\circ-\Omega_{ijk}\Phi^{ij}\right)\right]=0,\quad [(J^{ij}n_t+Q^{kij}n_k^\circ)\,n_i]=0$$

where, as usual, the square brackets [ ] denote jumps in the medium characteristics. Furthermore, assuming $\delta\xi^k\neq 0$, $\delta t\neq 0$ and taking account of the total variation formula

$$\{\delta u^\omega\}=\delta u^\omega-v^\omega\delta t-\frac{\partial u^\omega}{\partial\xi^k}\delta\xi^k$$

we will obtain still another relationship

$$\left[(L\sqrt{g}-J_\omega v^\omega)\,n_t+\frac{\partial\xi^k}{\partial x^\omega}\left(\sqrt{g}p^j{}_k n_j^\circ-\Omega_{ijk}\Phi^{ij}\right)v^\omega-\Phi^{ij}n_j\frac{\partial v_i{}^\wedge}{\partial n}\right]=0$$

The relationships obtained for the $\delta\xi^k$ are satisfied identically because of the equations for δt, as well as the known conditions of kinematic compatibility [17].

If the mass coservation equation is added to the obtained relationships, the complete system of conditions on the discontinuity of considered type will then be

$$[\rho\sqrt{g}]=0 \qquad (3.5)$$

$$\left[J_\omega n_t-\frac{\partial\xi^k}{\partial x^\omega}\left(\sqrt{g}p^j{}_k-\Omega_{ijk}\Phi^{ij}\right)\right]=0 \qquad (3.6)$$

$$[(J^{ij}n_t+Q^{kij}n_k^\circ)n_i]=0 \qquad (3.7)$$

$$\left[(L\sqrt{g}-J_\omega v^\omega)\,n_t+\frac{\partial\xi^k}{\partial x^\omega}\left(\sqrt{g}p^j{}_k n_j^\circ-\Omega_{ijk}\Phi^{ij}\right)v^\omega-\Phi^{ij}n_j\frac{\partial v_i{}^\wedge}{\partial n}\right]=0 \qquad (3.8)$$

Formula (3.6) may be considered as the momentum equation; (3.8) as the energy equation; (3.7) as additional "moment" relationships because of the presence of higher derivatives. Here $-n_t/|n^\circ|$ yields the propagation "velocity" of the surface of discontinuity in the ξ^k system. Relationships of another kind are obtained if the normal derivatives $\partial\delta u_j{}^\wedge/\partial n$ on both sides of the surface of discontinuity are considered independent.

Conditions (3.7) are then replaced by the following:

$$(J^{ij}n_t+Q^{kij}n_k^\circ)\,n_i|_+=0,\qquad (J^{ij}n_t+Q^{kij}n_k^\circ)\,n_i|_-=0$$

and (3.8) will have the simpler form

*) The considered surface is not closed, but the integration is easily extended to a closed surface consisiting of the surface of discontinuity and the surface of the body since the variations $\delta u_j{}^\wedge$ are zero on the latter. Such a surface may evidently always be chosen smooth, which is essential since otherwise additional contour integrals will appear in the presented formula.

$$\left[(L\sqrt{g} - J_\omega v^\omega)\, n_l + \frac{\partial \xi^k}{\partial x^\omega}\,(\sqrt{g}\, p^j{}_k n_j^\circ - \Omega_{ijk}\Phi^{ij})\, v^\omega\right] = 0$$

The peculiarity of the obtained conditions is that they reflect the geometric properties of the surface of discontinuity (in terms of the Ω_{ijk}).

Let us note, in conclusion. that these conditions simplify greatly in the case of small deformations, and go over into the customary conditions on a shock [18] in the absence of higher derivatives.

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